Brownian motion and Stochastic Calculus
Dylan Possamaï

Assignment 6

Exercise 1

Let $(W_t)_{t\geq 0}$ be a Brownian motion. For any $a>0$ consider the random times

$$T_a := \inf\big\{t>0 : W_t \geq a\big\},\ \overline{T}_a := \inf\big\{t>0 : |W_t| \geq a\big\},$$

1) Show that these random times are $\mathbb{F}^{W,\mathbb{P}}$–stopping times.

2) Show that the Laplace transform of T_a has the value

$$\mathbb{E}^{\mathbb{P}}\big[\exp(-\mu T_a)\big] = \exp\big(-a\sqrt{2\mu}\big),\ \forall \mu>0,$$

and show that $\mathbb{P}[T_a<\infty]=1$.

Hint: consider the martingale $M_t^\lambda := \exp(\lambda W_t - \frac{\lambda^2}{2}t)$ and use the optional sampling theorem.

4) Show that the Laplace transform of $\overline{T}_a$ has the value

$$\mathbb{E}^{\mathbb{P}}\big[\exp(-\mu \overline{T}_a)\big] = \frac{1}{\cosh(a\sqrt{2\mu})},\ \forall \mu>0.$$

Exercise 2

Let W be a Brownian motion on $[0,\infty)$ and $S_0>0$, $\sigma>0$, $\mu\in\mathbb{R}$ constants. The stochastic process $S=(S_t)_{t\geq 0}$ given by

$$S_t := S_0\exp\big(\sigma W_t + (\mu-\sigma^2/2)t\big),$$

is called *geometric Brownian motion.*

1) Prove that for $\mu\neq\sigma^2/2$, we have

$$\lim_{t\to\infty} S_t = +\infty,\ \mathbb{P}\text{–a.s., or } \lim_{t\to\infty} S_t = 0,\ \mathbb{P}\text{–a.s.}$$

When do the respective cases arise?

2) Discuss the behaviour of S_t as $t\longrightarrow\infty$ in the case $\mu=\sigma^2/2$.

3) For $\mu=0$, show that S is a martingale, but not uniformly integrable.

Exercise 3

Let B be a standard Brownian motion. Let $S^\star\in[0,1]$ be the smallest $s\in[0,1]$ with $B_s=\sup_{t\in[0,1]}B_t$. Moreover, let $L:=\sup\{t\in[0,1]\colon B_t=0\}$ be the last time in the interval $[0,1]$ when B is at 0.

1) Show that $\mathbb{P}$–a.s., B attains its maximum on the interval $[0,1]$ at a unique point.

2) Let W be a standard Brownian motion, independent of B. Prove that whenever $s\in[0,1]$, we have

$$\mathbb{P}[S^\star<s] = \mathbb{P}\bigg[\sup_{t\in[0,s]}B_t > \sup_{t\in[0,1-s]}W_t\bigg].$$

3) Let N and N' be random variables distributed as $N(0,1)$ and independent. Show that

$$\mathbb{P}[S^\star < s] = \mathbb{P}\big[\sqrt{s}|N| > \sqrt{1-s}|N'|\big] = 2\arcsin(\sqrt{s})/\pi.$$

The law of $S^\star$ is called the Arcsine distribution.

4) Show also that

$$\mathbb{P}[L < s] = \mathbb{P}\Big[\sup_{t\in[0,s]} B_t > \sup_{t\in[0,1-s]} W_t\Big], \text{ for } s \in [0,1],$$

so that L and $S^\star$ have the same law.

Exercise 4

Let $(B_t)_{t\geq 0}$ be a Brownian motion and $M_t := sup_{s\leq t} B_s$. Show that the joint distribution of the pair (B_t, M_t) is absolutely continuous with density

$$f_t(x,y) := \frac{2(2y-x)}{\sqrt{2\pi t^3}} \exp\Big(-\frac{(2y-x)^2}{2t}\Big)\mathbf{1}_{\{y\geq 0\}}\mathbf{1}_{\{x\leq y\}},\ (x,y)\in\mathbb{R}^2.$$

Hint: Show that

(i) for $y > 0$, $x \leq y$, $\mathbb{P}[B_t \leq x,\ M_t \geq y] = \mathbb{P}[B_t \geq 2y - x]$;

(ii) for $y > 0$, $x \leq y$,

$$\mathbb{P}[B_t \leq x,\ M_t \leq y] = \Phi\Big(\frac{x}{\sqrt{t}}\Big) - \Phi\Big(\frac{x-2y}{\sqrt{t}}\Big),$$

where Φ is the distribution function of a standard Gaussian random variable;

(iii) for $y > 0$, $x \geq y$,

$$\mathbb{P}[B_t \leq x,\ M_t \leq y] = \mathbb{P}[M_t \leq y] = \Phi\Big(\frac{y}{\sqrt{t}}\Big) - \Phi\Big(-\frac{y}{\sqrt{t}}\Big),$$

and for $y \leq 0$, $\mathbb{P}[B_t \leq x,\ M_t \leq y] = 0$.